3. Fluid Dynamics as a Mathematical Science

The 3rd of 12 lectures by Prof. Raj to share his perspective on effective application of computational aerodynamics to aircraft design.

Each lecture contains excerpts from the presentation shown below describing his exciting journey on a long and winding road for more than five decades!

Reflections on the Effectiveness of Applied Computational Aerodynamics for Aircraft Design

https://www.aoe.vt.edu/people/emeritus/raj/personal-page/reflections-on-ACA-effectiveness.html

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Advanced Development Programs

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Lecture 2: Key Takeaways

- **Early Days of Civilization**
  - **Two sets of "Grand Challenge" Problems**
    1. Problems of **Resistance** (ships, water wheels, projectiles,…)
    2. Problems of **Discharge** (water distribution, jet reaction machines,…)
  - **Two Branches of Investigations**
    1. **Hydraulics** (artisan activity based on empirical knowledge)
    2. **Hydrodynamics** (scientific activity based on fundamental laws of nature)

- **Key Foundational Ideas for Fluid Dynamics *(Antiquity to 1750)***
  - **384-322 BC: Aristotle—*concept of continuum***
  - **287-212 BC: Archimedes—*principles of hydrostatics***
  - **1452-1519: Leonardo da Vinci—*principles of continuity and relative motion***
  - **1586: Stevin—*hydrostatic pressure depends only on the height of the fluid column***
  - **1644: Torricelli—*efflux velocity is proportional to the square root of depth***
  - **1669: Huygens—*resistance is proportional to <u>square</u> of velocity***
  - **1687: Newton—*Laws of Mechanics and theory of fluid resistance***
  - **1738: D. Bernoulli—*pressure decreases as velocity increases***
  - **1742: J. Bernoulli—*concept of internal pressure in moving fluids***
  - **1749: d'Alembert—*symmetrical body would suffer no fluid force--a Paradox!***

Topics

Mathematical Science

We define Mathematical Science as *the application of the concepts, operations, and procedures of mathematics to study scientific fields*.

The mathematical concepts of zero and infinity—originating in ancient India—are the foundational building blocks of modern analytical and digital computing methods for scientific studies!

- **"Like the crests on the heads of peacocks…mathematics is at the top of all branches of knowledge."** *– Lagadha in Jyotiṣa Vedāṅga (earliest astronomical text from India ca. 1400 BCE)*
- **"The mathematical sciences particularly exhibit order, symmetry, and limitations; and these are the greatest forms of the beautiful."** *– Aristotle (384–322 BCE)*
- **"The laws of nature are but the mathematical thoughts of God."** *– Euclid (325–265 BCE)*
- **"Mathematics is the gate and key to science."** *– Bacon (1267)*
- **"No human investigation can be called real science if it cannot be demonstrated mathematically."** *– da Vinci (1452–1519)*
- **"Mathematics is a more powerful instrument of knowledge than any other that has been bequeathed to us by human agency." –** *Descartes (1596-1650)*
- **"A science is exact only insofar as it employs mathematics."** *– Kant (1724–1804)*
- **"Mathematics is the queen of the sciences."** *– Gauss (1777–1855)*
- **"A physical law must possess mathematical beauty."** *– Dirac (1902–1984)*

Fluid Dynamics as a Mathematical Science (1750 – 1900)

The Euler Equations (1755-57)

Leonhard Euler

15 Apr 1707 – 8 Sep 1783

$$P - \frac{1}{q}\left(\frac{dp}{dx}\right) = \left(\frac{du}{dt}\right) + u\left(\frac{du}{dx}\right) + v\left(\frac{du}{dy}\right) + w\left(\frac{du}{dz}\right)$$

$$Q - \frac{1}{q}\left(\frac{dp}{dy}\right) = \left(\frac{dv}{dt}\right) + u\left(\frac{dv}{dx}\right) + v\left(\frac{dv}{dy}\right) + w\left(\frac{dv}{dz}\right)$$

$$R - \frac{1}{q}\left(\frac{dp}{dz}\right) = \left(\frac{dw}{dt}\right) + u\left(\frac{dw}{dx}\right) + v\left(\frac{dw}{dy}\right) + w\left(\frac{dw}{dz}\right)$$

$$\left(\frac{dq}{dt}\right) + \left(\frac{d.qu}{dx}\right) + \left(\frac{d.qv}{dy}\right) + \left(\frac{d.gw}{dz}\right)^{*} = 0$$

equation of state *(a relation between p, q and r)*

*misprint: g should be q.

"…it is not the laws of Mechanics that we lack…but only the Analysis, which has not yet been sufficiently developed…"

"…steady direct motion in round tubes is stable or unstable according as $\rho DU_m/\mu$ <1900 or >2000,…"

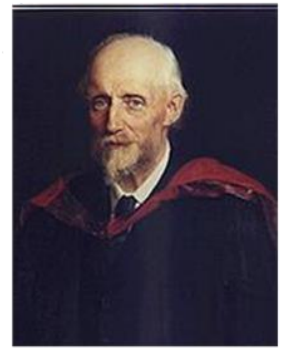
$$\rho\frac{d\bar{u}}{dt} = -\left\{\frac{d}{dx}\left(\overline{p_{xx}} + \rho\bar{u}\bar{u} + \rho\overline{u'u'}\right) + \frac{d}{dy}\left(\overline{p_{yx}} + \rho\bar{u}\bar{v} + \rho\overline{u'v'}\right) + \frac{d}{dz}\left(\overline{p_{zx}} + \rho\bar{u}\bar{w} + \rho\overline{u'w'}\right)\right\}$$

&c.

&c.

"…equations of mean-mean-motion…"

Osborne Reynolds

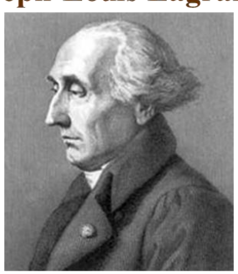


23 Aug 1842 – 21 Feb 1912

The Reynolds-Averaged Navier-Stokes (RANS) Equations (1895)

1750 • ***1800*** • ***1850*** • ***1900***

Joseph-Louis Lagrange



25 Jan 1736 – 10 Apr 1813

steady incompressible flow

$$\frac{\rho v^2}{2} + p = \text{const}$$

Bernoulli's Equation (1778)

George Stokes

13 Aug 1819 – 1 Feb 1903

The Navier-Stokes Equations (1849)

$$\rho\left(\frac{Du}{Dt} - X\right) + \frac{dp}{dx} - \mu\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right) - \frac{\mu}{3}\frac{d}{dx}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0,$$

$$\frac{d\rho}{dt} + \frac{d\rho u}{dx} + \frac{d\rho v}{dy} + \frac{d\rho w}{dz} = 0,$$

equation connecting p and ρ,

…conditions which must be satisfied at the surface of a solid in contact with the fluid…are unknown.

Source: Refs. 3.1 – 3.7; Wikipedia

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Foundations of Mathematical Fluid Dynamics L3

(18th Century)

'PRINCIPES GÉNÉRAUX DU MOUVEMENT DES FLUIDES'

Académie Royale des Sciences et des Belles-Lettres de Berlin

Presented 4 September 1755 [printed in 1757]

- **Three *equations of motion* derived from the *first axioms of mechanics* using *'infinitesimal fluid particle'***

$$P - \frac{1}{q}\left(\frac{dp}{dx}\right) = \left(\frac{du}{dt}\right) + u\left(\frac{du}{dx}\right) + v\left(\frac{du}{dy}\right) + w\left(\frac{du}{dz}\right)$$

$$Q - \frac{1}{q}\left(\frac{dp}{dy}\right) = \left(\frac{dv}{dt}\right) + u\left(\frac{dv}{dx}\right) + v\left(\frac{dv}{dy}\right) + w\left(\frac{dv}{dz}\right)$$

$$R - \frac{1}{q}\left(\frac{dp}{dz}\right) = \left(\frac{dw}{dt}\right) + u\left(\frac{dw}{dx}\right) + v\left(\frac{dw}{dy}\right) + w\left(\frac{dw}{dz}\right)$$

P, Q, R: accelerative forces due to gravity

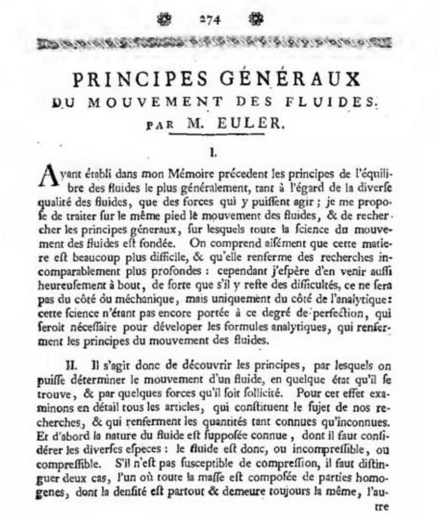
p, q, u, v, w: pressure, density, and three components of velocity

( ): partial derivatives

- **One *continuity equation***

$$\left(\frac{dq}{dt}\right) + \left(\frac{d.qu}{dx}\right) + \left(\frac{d.qv}{dy}\right) + \left(\frac{d.gw}{dz}\right)^{*} = 0$$

- **One *equation (now called equation of state)*:** relation between pressure, *p*, density, *q*, and another property, *r*, [temperature] which, in addition to *q*, influences *p* in compressible fluid (nature of fluid is assumed to be known.)



274

PRINCIPES GÉNÉRAUX
DU MOUVEMENT DES FLUIDES.
PAR M. EULER.

I.

Ayant établi dans mon Mémoire précedent les principes de l'équilibre des fluides le plus généralement, tant à l'égard de la diverſe qualité des fluides, que des forces qui y puiſſent agir ; je me propoſe de traiter ſur le même pied le mouvement des fluides, & de rechercher les principes géneraux, ſur lesquels toute la ſcience du mouvement des fluides eſt fondée. On comprend aiſément que cette matiere eſt beaucoup plus difficile, & qu'elle renferme des recherches incomparablement plus profondes : cependant j'eſpère d'en venir auſſi heureuſement à bout, de ſorte que s'il y reſte des difficultés, ce ne ſera pas du côté du méchanique, mais uniquement du côté de l'analytique: cette ſcience n'étant pas encore portée à ce degré de perfection, qui ſeroit néceſſaire pour déveloper les formules analytiques, qui renferment les principes du mouvement des fluides.

II. Il s'agit donc de découvrir les principes, par lesquels on puiſſe déterminer le mouvement d'un fluide, en quelque état qu'il ſe trouve, & par quelques forces qu'il ſoit ſollicité. Pour cet effet examinons en détail tous les articles, qui conſtituent le ſujet de nos recherches, & qui renferment les quantités tant connues qu'inconnues. Et d'abord la nature du fluide eſt ſuppoſée connue, dont il faut conſidérer les diverſes eſpeces : le fluide eſt donc, ou incompreſſible, ou compreſſible. S'il n'eſt pas ſuſceptible de compreſſion, il faut diſtinguer deux cas, l'un où toute la maſſe eſt compoſée de parties homogenes, dont la denſité eſt partout & demeure toujours la même, l'autre

Leonhard Euler

Swiss Mathematician
15 Apr 1707 – 8 Sep 1783

"…five equations encompassing the entire theory of the motion of fluids." — Euler

*misprint in original paper: *g* should be *q*

Source: Refs. 3.1 – 3.3, and Wikipedia

Tokaty on Euler's Equations

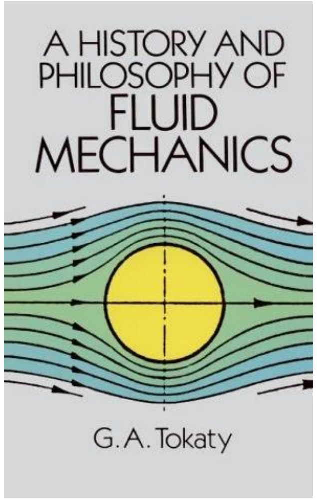
G.A. Tokaty, Soviet Scientist, Zhukovsky Academy (defected to Britain in 1947)

Emeritus Professor, Aeronautics and Space Technology, The City University, London

"…geometry is a branch of mathematics which treats the shape and size of things; while Fluidmechanics is the science of motion (and equilibrium) of bodies of deformable (and variable) shapes, under the action of forces…**some theorems and axioms of geometry do not meet the philosophical and physical needs of mechanics generally, and of Fluidmechanics in particular**… For example, a point is usually defined as an element of geometry which has position but no extension; a line as a path traced out by a point in motion…But motion and matter cannot be divorced. A **point** that has no extension lacks volume and, consequently, mass, therefore **is nothing**; and **nothing can have neither path nor momentum, or motion.**"

Grigori Tokaty

13 Oct 1909 – 23 Nov 2003



"Euler was, perhaps, the first to overcome this fundamental contradiction, by means of the introduction of his historic 'fluid particle', thus giving Fluidmechanics a powerful instrument of physical and mathematical analysis."

Euler imagined a fluid particle as an infinitesimal body, small enough to be treated mathematically as a point, but large enough to possess such physical properties as volume, mass, density, inertia, etc.

"The Blood, the Flesh, and the Bones of Fluid Mechanics"

 Note: Highlighting is by Raj

Source: Ref. 2.5

Euler's Observations on His Five Equations of Motion of Fluids

'PRINCIPES GÉNÉRAUX DU MOUVEMENT DES FLUIDES' 4 Sep 1755 [printed in 1757]

- "The equations contain four variables x, y, z and t which are absolutely independent of each other… the other variables u, v, w, p and q must be certain functions of the former."
- "…before we can begin to solve the equations, we need to know what sort of functions of x, y, z and t must be used to express the values of u, v, w, p and q …"
- "However, since very little work has yet been done…*we cannot hope to obtain a complete solution of our equations until the limits of Analysis have been extended much further*."
- "The best approach would therefore be to ponder well on the particular solutions of our differential equation that we are in a position to obtain…"
- "…if the three velocities are known, we can determine the trajectory described by each element of the fluid in motion." [streamlines]
- "If the shape of the vessel in which the fluid moves is given, the fluid particles that touch the surface of the vessel must necessarily follow its direction,…" [surface boundary condition]

"…it is not the laws of Mechanics that we lack…but only the Analysis, which has not yet been sufficiently developed for this purpose. It is therefore clearly apparent what discoveries we still need to make in this branch of Science before we can arrive at a more perfect Theory of the motion of fluids."

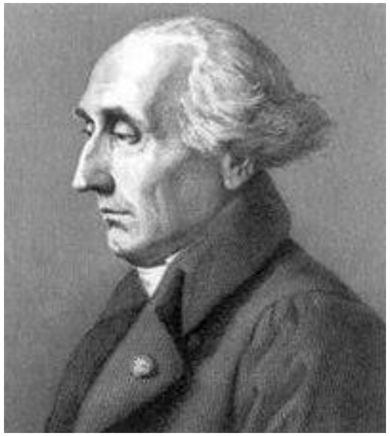
Source: Refs. 3.1 – 3.3

Analytical Solutions of Euler Equations

(18th Century)

Lagrange (1778) matured *'total differential' notion* into a powerful mathematical tool and applied it to Euler equations to conclude: "*the equations could be solved only for two particular cases*"

Joseph-Louis Lagrange



Franco-Italian Mathematician
25 Jan 1736 – 10 Apr 1813

1. **Unsteady Compressible Flow**

   By *introducing velocity potential, $\varphi\,(x,y,z,t)$*, and *gravitational potential, $\Phi\,(x,y,z)$*, Lagrange reduces Euler equations to a single total differential equation whose integral is

$$\frac{v^2}{2} + \int \frac{dp}{\rho} + \frac{\partial \varphi}{\partial t} - d\Phi = C(t)$$

2. **Steady Compressible Flow**

   Solution is the equation for case 1 (above) subject to $\partial\varphi/\partial t = 0$, and $C(t)$ just a constant.

   - For **steady, incompressible flows**, the solution of the Euler equations is

$$\frac{v^2}{2} + \frac{p}{\rho} - \Phi = C = \text{const}$$

The third term is typically negligibly small compared to the first two, and we get the now widely known '<u>Bernoulli's Equation</u>'

$$\frac{\rho v^2}{2} + p = \text{const}$$

Lagrange's Concept of Velocity Potential Revolutionized Evolution of Fluid Dynamics—It Remains a Vital Part to This Day

Source: Ref. 2.5 and Wikipedia

Mathematical Underpinnings of Potential Flow Theory

(18th Century)

- **Scalar Potential**
  - Scalar potential is the scalar value associated with every point in a field.
  - It's a **fundamental mathematical concept** that simplifies the study of quantities whose definition requires both magnitude and direction (vectors) over a given field or domain. *Beware that all vector fields do not have scalar potential!*
  - In physics, it describes the situation where the difference in the potential energies of an object at two locations depends only on its location, not upon the path taken; examples include gravitational potential and electrostatic potential
  - In an orthogonal coordinate system, partial derivatives of the potential give the magnitude of the vector
- **Potential Theory**
  - **Laplace (1783)** applied the language of calculus to show that a scalar potential, $V(x,y,z)$, always satisfies the differential equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \qquad \nabla^2 V = 0$$

Laplace's Equation

  - Mathematicians developed many methods to solve this linear, second-order PDE subject to prescribed boundary conditions

Pierre-Simon Laplace

French Scholar
23 Mar 1749 – 5 Mar 1827

"All the Effects of Nature are only the Mathematical Consequences of a Small Number of Immutable Laws." — Laplace

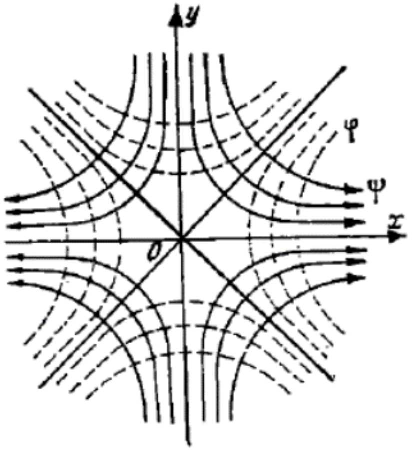
Source: Ref. 2.5 and Wikipedia

Advances in Fluid Dynamics:
Driven by Mathematical Techniques
(19th Century)

Augustin-Louis Cauchy

French Mathematician
21 Aug 1789 – 23 May 1857

- **Cauchy (1841) mathematically proved that motion of a fluid particle consists of three parts**
  a. Translational motion at velocity $\boldsymbol{V}\ (v_x, v_y, v_z)$
  b. Rigid Body Rotational motion with angular velocity $\boldsymbol{\omega}\ (\omega_x\ \omega_y\ \omega_z)$
  c. Deformational motion characterized by function $\boldsymbol{\Phi}\ (x, y, z)$ with nine numbers representing rate of normal and shear strains
- **When $\boldsymbol{\omega}$ is zero, the flow is *irrotational* consisting of translational and deformational motions only; the vorticity of the fluid is zero**
- **For 2D, steady, incompressible, irrotational flow, Cauchy showed that the stream function, $\psi(x,y)$, too satisfied Laplace's equation, much like the velocity potential, $\varphi(x,y)$**
  - $\varphi(x,y)$ and $\psi(x,y)$, are associated through the Cauchy-Riemann conditions, and are called conjugate functions
  - Fluid flows can be represented by equipotential (φ = *const.) lines* and streamlines (ψ = *const.*) that are orthogonal
  - Associated theory of *analytic functions of complex variables* offers many interesting and important solutions



Source: Ref. 2.5, 3.4 and Wikipedia

Green's Theorem

A Key Theorem for Mathematical Analysis of Potential Flows

(19th Century)

AN ESSAY ON THE APPLICATION OF MATHEMATICAL ANALYSIS TO THE THEORIES OF ELECTRICITY AND MAGNETISM

Originally published as a book in Nottingham, 1828.

Reprinted in three parts in Journal für die reine und angewandte Mathematik Vol. 39, 1 (1850) p. 73–89; Vol. 44, 4 (1852) p. 356–74; and Vol. 47, 3 (1854) p. 161–221. From there transcribed by Ralf Stephan (ralf@ark.in-berlin.de)

Before proceeding to make known some relations which exist between the density of the electric fluid at the surfaces of bodies, and the corresponding values of the potential functions within and without those surfaces, the electric fluid being confined to them alone, we shall in the first place, lay down a general theorem which will afterwards be very useful to us. This theorem may be thus enunciated:

Let U and V be two continuous functions of the rectangular co-ordinates x, y, z, whose differential co-efficients do not become infinite at any point within a solid body of any form whatever; then will

$$\int dx\, dy\, dz\, U\delta V + \int d\sigma U\left(\frac{dV}{dw}\right) = \int dx\, dy\, dz\, V\delta U + \int d\sigma V\left(\frac{dU}{dw}\right);$$

the triple integrals extending over the whole interior of the body, and those relative to $d\sigma$, over its surface, of which $d\sigma$ represents an element: dw being an infinitely small line perpendicular to the surface, and measured from this surface towards the interior of the body.

Note that: $\delta V = \dfrac{d^2V}{dx^2} + \dfrac{d^2V}{dy^2} + \dfrac{d^2V}{dz^2}$

George Green

British Mathematician
14 Jul 1793 – 31 May 1841

Source: Ref. 3.5 & 3.6 and Wikipedia

Ideal Fluid Dynamics:

Application of Green's Theorem to Irrotational Flows

(19th Century)

Extensions of Green's theorem to ideal fluid dynamics followed naturally due to the analogy of velocity potential, ϕ, with electrostatic potential, magnetic potential, etc. (Lamb: *Treatise on the Mathematical Theory of the Motion of Fluids,* 1879; *Hydrodynamics,* 1895, 6th ed. 1932)

- If we denote the two continuous, single-valued functions, U and V, in the Green's theorem by ϕ and ϕ' *respectively,* each satisfying $\nabla^2\phi = 0$ and $\nabla^2\phi' = 0$ throughout a given region bounded by the surface S, then

$$\iint \phi \frac{\partial \phi'}{\partial n}\, dS = \iint \phi' \frac{\partial \phi}{\partial n}\, dS$$

- Taking ϕ to be the velocity potential and choosing $\phi' = 1/r$, the velocity potential ϕ_P at any point P in the space occupied by the fluid may be written as:

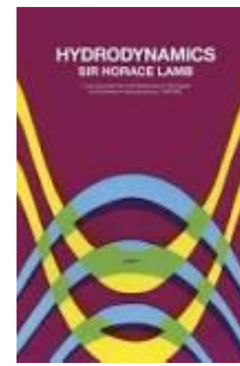
$$\phi_P = -\frac{1}{4\pi}\iint \frac{1}{r}\frac{\partial \phi}{\partial n}\, dS + \frac{1}{4\pi}\iint \phi \frac{\partial}{\partial n}\left(\frac{1}{r}\right) dS.$$

Only surface integrals!

1st term is surface distribution of simple sources with density $-\,\partial\phi/\partial n$, and 2nd term of double sources with axes normal to the surface and density ϕ. *This is only one of infinite surface distributions that give the same value of ϕ throughout the interior.*

Horace Lamb

British Mathematician
27 Nov 1849 – 4 Dec 1934



Dover edition, 1945
(republication of 1932 6th edition)

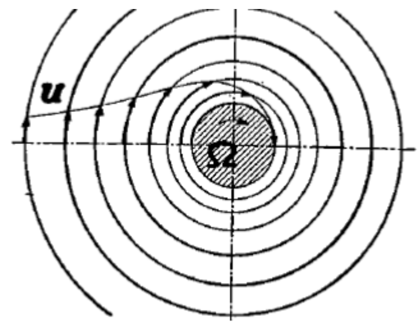
- The irrotational flow of fluids in a simply-connected region is determined when either ϕ or inward normal velocity $-\,\partial\phi/\partial n$ is prescribed at all points of the boundary, or ϕ over part of the boundary and $-\,\partial\phi/\partial n$ over the remainder.
- Lamb (*Ch. III, 6th ed. 1932*) shows that representations of ϕ_P in terms of simple sources *alone*, or of double sources *alone*, are unique.

Source: Ref. 3.5 & 3.6 and Wikipedia

Ideal Fluid Dynamics:

Key Theorems for Flows with Vorticity (Rotational Flows) (19th Century)

- **Helmholtz postulated three theorems (1858) based on his proof of indestructability and uncreatability of vorticity in inviscid, barotropic* fluid subjected to conservative body forces only**
  1. The strength of a vortex filament is constant along its length.
  2. A vortex filament cannot end in a fluid; it must extend to the boundaries of the fluid or form a closed path.
  3. In the absence of rotational external forces, a fluid that is initially irrotational remains irrotational.



Induced velocity field of a vortex filament

[Cauchy had mathematically proven (1841) that the motion of a fluid particle consisted of translational, rigid body rotational, and deformational motions; when rotational motion is not zero, the flow contains a string of rotating elements or vortex lines.]

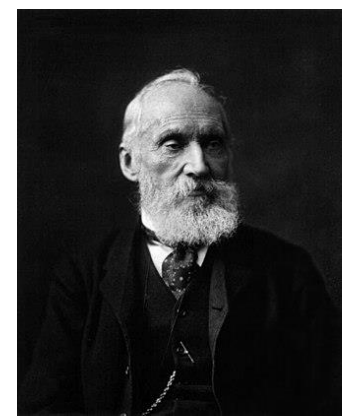
- **Kelvin Circulation Theorem (1867)**
  - Circulation (Γ) around a closed curve moving with the fluid remains constant with time, that is, $D\Gamma/Dt = 0$

*density is a function of only pressure

Hermann von Helmholtz

German Scientist & Philosopher
31 Aug 1821 – 8 Sep 1894

William Thomson
1st Baron Kelvin



British Mathematical Physicist
26 Jun 1824 – 17 Dec 1907

Source: Ref. 2.5, 3.7, 3.8 and Wikipedia

Modified Euler Equations

(19th Century)

Claude Louis Marie Henri Navier

French Engineer
10 Feb 1785 – 21 Aug 1836

Mémoire sur les lois du Mouvement des Fluides (1823)

Mémoires de l'Académie Royale des Sciences de l'Institut de France

- Contains modified Euler equations for incompressible flow based on a different model of fluid to account for ***attractive and repulsive intermolecular forces***

$$P - \frac{dp}{dx} = \rho\left(\frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz}\right) - \varepsilon\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right)$$

$$Q - \frac{dp}{dy} = \rho\left(\frac{dv}{dt} + u\frac{dv}{dx} + v\frac{dv}{dy} + w\frac{dv}{dz}\right) - \varepsilon\left(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}\right)$$

$$R - \frac{dp}{dz} = \rho\left(\frac{dw}{dt} + u\frac{dw}{dx} + v\frac{dw}{dy} + w\frac{dw}{dz}\right) - \varepsilon\left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2}\right)$$

- ε is a function of spacing between molecules

- **Slip boundary condition**: e.g., for a wall perpendicular to z-axis

$$Eu + \varepsilon\frac{du}{dz} = 0, \quad Ev + \varepsilon\frac{dv}{dz} = 0$$

- ε, a function of nature of fluid and wall, is to be determined experimentally

Navier's Modified Euler Equations Resemble Those for Viscous Fluids Derived by Stokes Based on His Theory of Internal Friction

Source: Ref. 3.9, and Wikipedia

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Theory of Viscous Fluids in Motion

(19th Century)

On the Theories of the Internal Friction of Fluids in Motion and of the Equilibrium and Motion of Elastic Solids, Transactions of Cambridge Philosophical Society, Vol. 8, pp 287-319, 1849 (Read April 14, 1845)

"The equations of Fluid Motion commonly employed depend upon *the fundamental hypothesis that the mutual action of two adjacent elements of the fluid is normal to the surface which separates them.*"

XXII. *On the Theories of the Internal Friction of Fluids in Motion, and of the Equilibrium and Motion of Elastic Solids. By* G. G. Stokes, M.A., *Fellow of Pembroke College.*

[Read *April* 14, 1845.]

The equations of Fluid Motion commonly employed depend upon the fundamental hypothesis that the mutual action of two adjacent elements of the fluid is normal to the surface which separates them. From this assumption the equality of pressure in all directions is easily deduced, and then the equations of motion are formed according to D'Alembert's principle. This appears to me the most natural light in which to view the subject; for the two principles of the absence of tangential action, and of the equality of pressure in all directions ought not to be assumed as independent hypotheses, as is sometimes done, inasmuch as the latter is a necessary consequence of the former*. The equations of motion so formed are very complicated, but yet they admit of solution in some instances, especially in the case of small oscillations. The results of the theory agree on the whole with observation, so far as the time of oscillation is concerned. But there is a whole class of motions of which the common theory takes no cognizance whatever, namely, those which depend on the tangential action called into play by the sliding of one portion of a fluid along another, or of a fluid along the surface of a solid, or of a different fluid, that action in fact which performs the same part with fluids that friction does with solids.

Thus, when a ball pendulum oscillates in an indefinitely extended fluid, the common theory gives the arc of oscillation constant. Observation however shows that it diminishes very rapidly in the case of a liquid, and diminishes, but less rapidly, in the case of an elastic fluid. It has indeed been attempted to explain this diminution by supposing a friction to act on the ball, and this hypothesis may be approximately true, but the imperfection of the theory is shown from the circumstance that no account is taken of the equal and opposite friction of the ball on the fluid.

Again, suppose that water is flowing down a straight aqueduct of uniform slope, what will be the discharge corresponding to a given slope, and a given form of the bed? Of what magnitude must an aqueduct be, in order to supply a given place with a given quantity of water? Of what form must it be, in order to ensure a given supply of water with the least expense of materials in the construction? These, and similar questions are wholly out of the reach of the common theory of Fluid Motion, since they entirely depend on the laws of the transmission of that tangential action which in it is wholly neglected. In fact, according to the common theory the water ought to flow on with uniformly accelerated velocity; for even the supposition of a certain friction against the bed would be of no avail, for such friction could not be transmitted through the mass. The practical importance of such questions as those above mentioned has made them the object of numerous experiments, from which empirical formulæ have been constructed. But such formulæ, although fulfilling well enough the purposes for which they were

* This may be easily shown by the consideration of a tetrahedron of the fluid, as in Art. 4.

George Stokes

British Mathematician & Physicist
13 Aug 1819 – 1 Feb 1903

"But *there is a whole class of motions* of which the common theory takes no cognizance whatever, namely, those *which depend on the tangential action called into play by the sliding of one portion of a fluid along another, or of a fluid along the surface of a solid*, or of a different fluid, that action in fact which performs the same part with fluids that friction does with solids."

"Again, suppose that water is flowing down a straight aqueduct of uniform slope, what will be the discharge corresponding to a given slope, and a given form of the bed? Of what magnitude must an aqueduct be, in order to supply a given place with a given quantity of water? Of what form must it be, in order to ensure a given supply of water with the least expense of materials in the construction? *These, and similar questions are wholly out of the reach of the common theory of Fluid Motion, since they entirely depend on the laws of the transmission of that tangential action which in it is wholly neglected.*"

Source: *Ref. 3.10, and Wikipedia*

The Navier-Stokes Equations

(19th Century)

George Stokes

British Mathematician & Physicist
13 Aug 1819 – 1 Feb 1903

On the Theories of the Internal Friction of Fluids in Motion and of the Equilibrium and Motion of Elastic Solids
Transactions of Cambridge Philosophical Society, Vol. 8, pp 287-319, 1849 (Read April 14, 1845)

$$\frac{d\rho}{dt} + \frac{d\rho u}{dx} + \frac{d\rho v}{dy} + \frac{d\rho w}{dz} = 0, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (11)$$

$$\rho\left(\frac{Du}{Dt} - X\right) + \frac{dp}{dx} - \mu\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right) - \frac{\mu}{3}\frac{d}{dx}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0, \ \&c. \ldots\ldots (12)$$

equation connecting p and ρ,

μ is assumed to be constant, not dependent on *pressure* or *temperature*

Boundary condition for fluid in contact with a solid

"The most interesting questions connected with this subject require for their solution a knowledge of the conditions which must be satisfied at the surface of a solid in contact with the fluid, which, *except perhaps in case of very small motions*, are unknown."

Source: Ref. 3.10, and Wikipedia

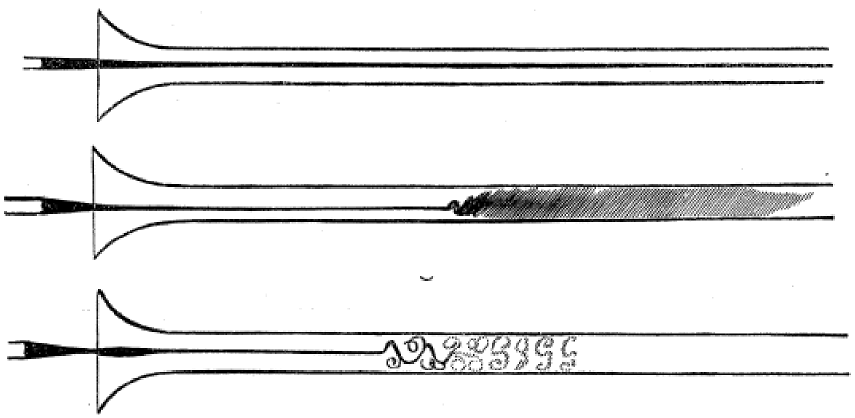
VT | College of Engineering — Kevin T. Crofton Department of Aerospace and Ocean Engineering — Virginia Tech

Distinct Types of Viscous Flows

(19th Century)

An Experimental Investigation of the Circumstances which determine whether the Motion of Water shall be Direct or Sinuous, and of the Law of Resistance in Parallel Channels

Philosophical Transactions of the Royal Society of London, 174, 1883, pp 935-982 (Read March 15, 1883)



Direct

With increasing $\rho c U_m/\mu$

Sinuous

Osborne Reynolds

British Engineer and Physicist
23 Aug 1842 – 21 Feb 1912

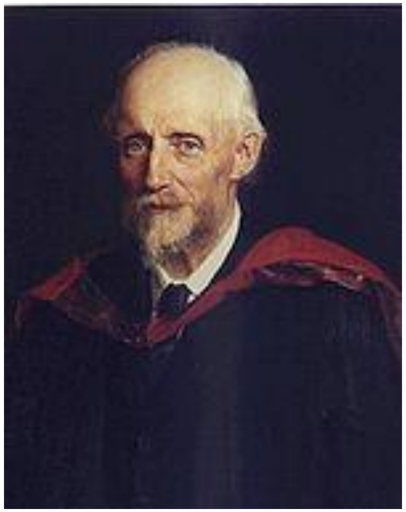
"…the broad fact of there being a critical value for the velocity [U_m] at which the steady motion becomes unstable, which critical value is proportional to $\mu/\rho c$ where c is the diameter of the pipe and μ/ρ the viscosity by the density, is abundantly established."

Source: Ref. 3.11, and Wikipedia

Governing Equations of Turbulent Flows

(19th Century)

Osborne Reynolds



British Engineer and Physicist
23 Aug 1842 – 21 Feb 1912

On the Dynamical Theory of Incompressible Viscous Fluids and the Determination of the Criterion, Philosophical Transactions of the Royal Society of London (A), 186, 1895, pp 123-164 (Read May 24, 1894)

- **Experimental criterion**: ***"…steady direct motion in round tubes is stable or unstable according as $\rho D U_m/\mu$ <1900 or >2000…a criterion for the possible maintenance of sinuous or eddying motion."***
- **Theoretical development**: **introduced concepts of '*mean-mean-motion*' and '*relative-mean-motion*'**

$$\bar{u} = \frac{\Sigma(\rho u)}{\Sigma \rho}, \text{ \&c., \&c.} \qquad (4).$$

$$\rho u = \rho \bar{u} + \rho u' \qquad (5)$$

- **Equations of mean-mean-motion of turbulent flows**

$$\rho \frac{d\bar{u}}{dt} = -\left\{ \frac{d}{dx}\left(\overline{p_{xx}} + \rho \bar{u}\bar{u} + \overline{\rho u' u'}\right) + \frac{d}{dy}\left(\overline{p_{yx}} + \rho \bar{u}\bar{v} + \overline{\rho u' v'}\right) + \frac{d}{dz}\left(\overline{p_{zx}} + \rho \bar{u}\bar{w} + \overline{\rho u' w'}\right) \right\} \qquad (15),$$

$$\text{\&c.} = \text{\&c.}$$

$$\text{\&c.} = \text{\&c.}$$

Reynolds stresses

The Reynolds-Averaged Navier-Stokes (RANS) Equations!

Source: Ref. 3.12, 3.13 and Wikipedia

Reynolds' 1895 Paper with RANS Equations

A Transformative Achievement!

- **Reynolds' Motivation for the 1895 Paper**
  - **Response to Lord Rayleigh's review comment on Reynolds' landmark 1883 paper:** *'In several places the author refers to theoretical investigation whose nature is not sufficiently indicated.'*
  - **In the 1895 paper, Reynolds offers proof of the existence of the criterion for the values of $K = \rho D U_m / \mu$ when direct motion changes to sinuous**
- **Expert Reviewer Comments on the Paper**
  - **Sir George Stokes:** *'…the author…himself considers it [paper] as of much importance. I confess I am not prepared to endorse that opinion myself, but neither can I say that it may not be true.'*
  - **Sir Horace Lamb:** *'…the paper should be published in the Transactions as containing the views of its author on a subject which he has to a great extent created, although much of it is obscure.'*
- **The "Closure Problem" needs to be solved for RANS equations to be usable**
  - *"…**one needs a means for determining the Reynolds stresses in terms of known or calculable quantities [mean flow]**…Reynolds himself only obliquely touched on this." – Launder (2015)*
- **Turbulence Modeling (determining Reynolds stresses) for RANS equations**
  - **G.I. Taylor (1915):** *"…to consider the disturbed motion of layers of air [in the atmosphere], we can take account of the eddies by introducing a coefficient of eddy viscosity…which we can express as $\frac{1}{2}\rho\left(\overline{w}d\right)$ where d is an average height through which an eddy moves before mixing with its surroundings, and $\overline{w}$ roughly represents the average vertical velocity…where w' is positive."*

For more than 100 years, quest for 'better' turbulence models has remained the *"holy grail"* of science!

"Indeed, its impact on all our lives is incalculable." — Launder

Source: Ref. 3.13 & 3.14

Lecture 3: Overarching Takeaway

"Leonhard Euler was not a contributor to, but the founder of, Fluidmechanics, its mathematical architect, its great river."

- Grigori Tokaty

13 Oct 1909 – 23 Nov 2003

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